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1990 Eur. J. Phys. 11 31

(http://iopscience.iop.org/0143-0807/11/1/005)

Eur. J. Phys. 11 (1990) 31–34. Printed in the UK

Lagrangians for simple systems with variable mass

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Received 16 December 1988, in final form 9 August 1989

Abstract. The Lagrangian formulation of classical mechanics and its applications figure prominently in the educational literature. Yet systems with variable mass are summarily excluded from this formulation and discussed in terms of Newtonian theory only. This omission is neither technically justified nor desirable from a pedagogical point of view, because it might suggest to the student that such systems are beyond the scope of the Lagrangian approach. Therefore, we show that the formalism can be readily extended to include those variable mass systems that are treated in the textbooks in a Newtonian fashion, and its application is illustrated by means of three instructive examples.

Zusammenfassung. Die meisten Lehrbücher tragen der Lagrangeschen Formulierung der Klassischen Mechanik und ihren Anwendungen gebührend Rechnung. Allerdings beschränken sie sich dabei durchwegs auf die Diskussion von Systemen mit konstanter Masse: solche mit veränderlicher Masse werden immer nur im Rahmen des Newtonschen Zuganges behandelt. Diese Einschränkung ist aber nicht nur technisch unnötig, sondern auch pädagogisch sehr ungünstig, da die Studenten daraus schließen könnten, solche Systeme seien dem Lagrangeschen Formalismus nicht zugänglich. Deshalb zeigen wir, daß sich der Formalismus unschwer auf alle jene Fälle mit variabler Masse ausdehnen läßt, die in den Lehrbüchern mit der Newtonschen Methode behandelt werden, und wenden ihn auf drei instruktive Beispiele an.

1. Introduction

Because of its far reaching relevance for various fields of theoretical physics, most mechanics texts devote a chapter or two to the Lagrangian formulation of classical mechanics. Some even start with Lagrangian dynamics (Landau and Lifshitz 1960, Kilmister 1967). Indeed, the Lagrangian formulation greatly facilitates a change of coordinates, it allows constraints to be most easily incorporated, symmetries to be systematically exploited using Noether's theorem, and it is the starting point for the Hamiltonian formulation, which, in turn, is the basis of the transition to a quantum mechanical description of the system under study.

On the other hand, most of these texts also contain a brief discussion of systems with variable mass like the motion of a rocket expelling exhaust gas at a constant rate. However, these problems invariably appear only in the Newtonian formulation. From the pedagogical point of view, this is an undesirable restriction, since it seems to suggest an unwarranted limitation of the otherwise very powerful Lagrangian formalism. It is true that in the literature there is an occasional mention of a Lagrangian for a particular variable-mass system (Ray 1979), but without giving its origin, let alone a systematic derivation of the Lagrangians for more general systems.

We shall therefore show in this paper how all standard textbook systems with variable mass may be readily incorporated into the Lagrangian formalism. In §2 we first review the derivation of the Lagrangian from the equation of motion, which, in §3, we specialise to simple systems with variable mass. In §4 we illustrate the formalism with three instructive examples.

2. Lagrangians from the equation of motion

As textbook systems with variable mass are always one dimensional, we also restrict ourselves to systems with one degree of freedom, in order to avoid unnecessary complications. For such systems Darboux (1894) showed almost a century ago, how Lagrangians $L(x, \dot{x}, t)$ may be associated with their equation of motion (see also Yan 1978, Leubner 1981),

$$0 = \ddot{x} - (1/m)F(x, \dot{x}, t). \qquad (1)$$

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In order that the Euler–Lagrange equation derived from $L(x, \dot{x}, t)$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \qquad (2)$$

implies (1), we require

$$L_x - \dot{x}L_{\dot{x}x} - L_{\dot{x}t} = (1/m)F(x, \dot{x}, t)L_{\dot{x}\dot{x}} \qquad (3)$$

which is obvious from rewriting (2) in the form

$$0 = \ddot{x}L_{\dot{x}\dot{x}} + \dot{x}L_{\dot{x}x} + L_{\dot{x}t} - L_x$$
$$= L_{\dot{x}\dot{x}}\left(\ddot{x} - \frac{L_x - \dot{x}L_{\dot{x}x} - L_{\dot{x}t}}{L_{\dot{x}\dot{x}}}\right).$$

(Here and in the following, subscripts are a shorthand for derivatives with respect to the corresponding variable.)

With a given force $F(x, \dot{x}, t)$, equation (3) is a linear partial differential equation in the three variables x, $\dot{x}$, and t for the Lagrangian $L(x, \dot{x}, t)$. Darboux's idea was to simplify (3) by differentiating this equation once more with respect to $\dot{x}$, with the result

$$(1/m)F(x, \dot{x}, t)\Lambda_{\dot{x}} + \dot{x}\Lambda_x + \Lambda_t = -(1/m)F_{\dot{x}}(x, \dot{x}, t)\Lambda, \qquad (4)$$

where the abbreviation

$$\Lambda = L_{\dot{x}\dot{x}} \qquad (5)$$

has been introduced and the order of partial derivatives has been suitably interchanged.

Along any orbit of the system, where F/m is related to $\ddot{x}$ according to (1), equation (4) may be written more compactly as

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = -\frac{1}{m}F_{\dot{x}}(x, \dot{x}, t)\Lambda. \qquad (6)$$

It has been demonstrated (Darboux 1894, Yan 1978, Leubner 1981) how the general solution of (6) can be constructed, provided the solution to (1) is already known, and we refer the interested reader to the references for the general case. However, there are a number of force functions $F(x, \dot{x}, t)$ corresponding to physical systems of considerable pedagogical interest and including in particular those appearing in textbooks in the context of variable-mass systems, for which a solution to (6) can be given without any prior information about the orbit.

For example, with the force on the right-hand side of the damped harmonic oscillator equation,

$$m\ddot{x} = -b\dot{x} - kx$$

equation (6) reads

$$\mathrm{d}\Lambda/\mathrm{d}t = (b/m)\Lambda$$

with a solution being

$$\Lambda(x, \dot{x}, t) = \Lambda_0 \exp[(b/m)t]. \qquad (7)$$

As another example consider a point mass falling subject to a frictional force of Newtonian type in a homogeneous gravitational field, with initial conditions such that the sign of $\dot{x}$ does not change,

$$m\ddot{x} = -mg - b\dot{x}^2.$$

This system gives rise to

$$\mathrm{d}\Lambda/\mathrm{d}t = (2b/m)\dot{x}\Lambda,$$

with a solution being

$$\Lambda(x, \dot{x}, t) = \Lambda_0 \exp[(2b/m)x]. \qquad (8)$$

Once Λ is known, the Lagrangian is found by integrating (5) twice with respect to $\dot{x}$, and by subjecting the result to the Euler–Lagrange equation (2) (Leubner 1981),

$$L = \int_{v_0}^{\dot{x}} (\dot{x} - v)\Lambda(x, v, t)\mathrm{d}v$$
$$+ \frac{1}{m}\int_{x_0}^{x} F(\xi, v_0, t)\Lambda(\xi, v_0, t)\mathrm{d}\xi + \frac{\mathrm{d}\Omega(x, t)}{\mathrm{d}t}. \qquad (9)$$

Note that the choice of the lower limits of integration, v_0 and x_0, has no relevance, since different choices change only the mechanical gauge function $\Omega = \Omega(x, t)$.

Thus, with $v_0 = x_0 = 0$, the Lagrangian corresponding to (7) is

$$L = \exp\left(\frac{b}{m}t\right)\left(\frac{\dot{x}^2}{2} - \frac{k}{m}\frac{x^2}{2}\right) + \frac{\mathrm{d}\Omega}{\mathrm{d}t}$$

while the one corresponding to (8) is

$$L = \exp\left(\frac{2b}{m}x\right)\left(\frac{\dot{x}^2}{2} - \frac{mg}{2b}\right) + \frac{\mathrm{d}\Omega}{\mathrm{d}t}$$

where in both cases the irrelevant constant Λ_0 has been set equal to unity.

3. Lagrangians for simple systems with variable mass

In order to apply the results of §2 to the construction of Lagrangians for one-dimensional systems with variable mass, we first require the corresponding equation of motion. Many mechanics texts (Goodman and Warner 1964, Burghes and Downs 1975, Griffiths 1985) derive this equation in the following manner. With respect to a chosen inertial frame of reference, the momentum of the body at time $t + \mathrm{d}t$ is $(m(t) + \mathrm{d}m)(\dot{x} + \mathrm{d}\dot{x})$. Without the action of an external force $F(x, \dot{x}, t)$, this momentum would be the momentum at time t increased by the momentum acquired during $\mathrm{d}t$ as a result of adding the mass $\mathrm{d}m$ with a velocity $u(x, \dot{x}, t)$. Hence, the difference between the two quantities must be equal to the change of momentum induced by the action of the external force,

$$[m(t) + \mathrm{d}m][\dot{x} + \mathrm{d}\dot{x}] - [m(t)\dot{x} + \mathrm{d}m\, u(x, \dot{x}, t)]$$
$$= F(x, \dot{x}, t)\mathrm{d}t$$

from which we find

$$m(t)\ddot{x} = F(x, \dot{x}, t) + \dot{m}(t)[u(x, \dot{x}, t) - \dot{x}]. \quad (10)$$

The first step in constructing a corresponding Lagrangian consists of finding a solution to the variable mass equivalent of equation (6), namely,

$$\frac{d\Lambda}{dt} = -\left(\frac{1}{m(t)}F_{\dot{x}}(x, \dot{x}, t) + \frac{\dot{m}(t)}{m(t)}(u_{\dot{x}} - 1)\right)\Lambda. \quad (11)$$

There are two special cases of variable-mass systems that are of particular interest. Case (i) occurs when mass is added to the body at a given rate with a given velocity $u(x, \dot{x}, t) = u(t)$, as in figure 1, where wind-driven raindrops are falling onto a swinging bucket. In this case $u_{\dot{x}} = 0$ and (11) reduces to

$$\frac{d\Lambda}{dt} = \left(\frac{\dot{m}(t)}{m(t)} - \frac{1}{m(t)}F_{\dot{x}}(x, \dot{x}, t)\right)\Lambda. \quad (12)$$

Once a solution Λ to this equation is known, the corresponding Lagrangian can be found from (9), with F in the second integral replaced by the right-hand side of (10).

Case (ii) occurs when mass is lost at a given rate with given *relative* velocity $u(x, \dot{x}, t) = \dot{x} + u_{rel}(t)$, as in the example of a rocket expelling gas. In this case $u_{\dot{x}} = 1$ and (11) reduces to

$$\frac{d\Lambda}{dt} = -\frac{1}{m(t)}F_{\dot{x}}(x, \dot{x}, t)\Lambda. \quad (13)$$

Again, once a solution Λ to this equation is known, the corresponding Lagrangian can be found from (9), with F in the second integral replaced by the right-hand side of (10).

4. Illustrative examples

4.1. Example: linearly damped, swinging bucket hit by slanted rain

Here we are dealing with case (i) (see figure 1). The equation of motion (10) becomes

$$m(t)\ddot{x} = -b\dot{x} - m(t)(g/l)x + (u_0 - \dot{x})\mu_0 \quad (14)$$

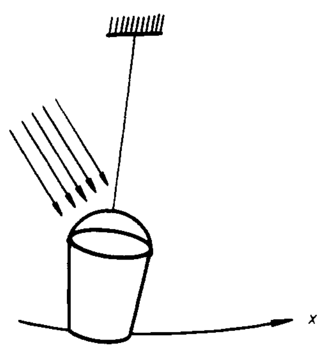


Figure 1. The swinging bucket hit by slanted rain.

where $\mu_0 = \dot{m}(t) > 0$ is the constant rate of mass increase, u_0 is the velocity of the raindrops in the x direction and l is the reduced pendulum length. Equation (12) thus becomes

$$\frac{d\Lambda}{dt} = \frac{b + \mu_0}{m_0 + \mu_0 t}\Lambda$$

with a solution being

$$\Lambda = (m_0 + \mu_0 t)^{1 + b/\mu_0}.$$

Substituting this into (9), we find the Lagrangian

$$L = (m_0 + \mu_0 t)^{1 + b/\mu_0}\left(\frac{\dot{x}^2}{2} - \frac{g}{l}\frac{x^2}{2} + \frac{u_0\mu_0}{m_0 + \mu_0 t}x\right) + \frac{d\Omega}{dt}. \quad (15)$$

It is a simple exercise to verify that (15) indeed implies the equation of motion (14).

4.2. Example: linearly damped, swinging bucket losing water at a constant rate through a hole in the base

This is case (ii) (see figure 2) with $u(x, \dot{x}, t) = \dot{x}$, that is, with $u_{rel} = 0$. The equation of motion (10) becomes

$$(m_0 - \mu_0 t)\ddot{x} = -b\dot{x} - (m_0 - \mu_0 t)(g/l)x$$
$$t < m_0/\mu_0 \quad (16)$$

with the same notations as before. Equation (13) for Λ reduces to

$$\frac{d\Lambda}{dt} = \frac{b}{m_0 - \mu_0 t}\Lambda$$

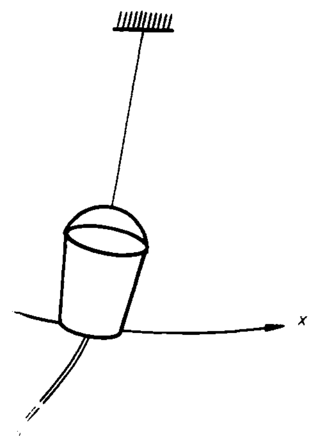
with a solution being

$$\Lambda = (m_0 - \mu_0 t)^{-b/\mu_0}.$$

Substituting this into (9), we find the corresponding Lagrangian

$$L = (m_0 - \mu_0 t)^{-b/\mu_0}\left(\frac{\dot{x}^2}{2} - \frac{g}{l}\frac{x^2}{2}\right) + \frac{d\Omega}{dt}. \quad (17)$$

Figure 2. The swinging bucket losing water.



It is again a simple exercise to verify that this Lagrangian implies the equation of motion (16).

4.3. Example: linearly damped, vertically ascending rocket

This is again case (ii), with $u(x, \dot{x}, t) = \dot{x} - u_{\text{rel}}(t)$, $u_{\text{rel}}(t) > 0$, $\dot{m} = -\mu_0$, $\mu_0 > 0$. The equation of motion (10) is in this case

$$(m_0 - \mu_0 t)\ddot{x} = -(m_0 - \mu_0 t)g - b\dot{x} + \mu_0 u_{\text{rel}}(t)$$
$$t < m_0/\mu_0 \tag{18}$$

with the same notation as in §4.2. Equation (13) now reads

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = \frac{b}{m_0 - \mu_0 t}\Lambda$$

with a possible solution being

$$\Lambda = (m_0 - \mu_0 t)^{-b/\mu_0}.$$

Inserting this into (9), the corresponding Lagrangian is found to be

$$L = (m_0 - \mu_0 t)^{-b/\mu_0}\left(\frac{\dot{x}^2}{2} - \frac{[(m_0 - \mu_0 t)g - \mu_0 u_{\text{rel}}(t)]x}{m_0 - \mu_0 t}\right) + \frac{\mathrm{d}\Omega}{\mathrm{d}t}. \tag{19}$$

Again, one can easily show that (19) implies the equation of motion (18).

5. Conclusions

In an analogous manner, a Lagrangian L can also be found for those other mechanical systems with variable mass that are treated in the Newtonian approach in the textbooks (Goodman and Warner 1964, Burghes and Downs 1975, Griffiths 1985) and which have not been included in §4.

Acknowledgments

One of the authors (C L) is grateful to his students B Gamper, A Hörtnagl and M Huber for asking the questions that prompted this investigation, and to the Department of Physics at the University of Natal for the hospitality extended to him.

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